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The response of a turbulent boundary layer to abrupt changes in surface conditions

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In a previous paper, it was shown that abrupt changes in the surface conditions under a very deep boundary layer cause changes of mean velocity and temperature that satisfy the dynamical conditions for self-preserving development. Here the theory is extended to predict the development of the modified flow in the moderately deep layers that occur in nature and the laboratory. The problems considered are the changes in the velocity profile produced by an abrupt change of surface roughness and also by a line of concentrated roughness such as a fence, the changes in temperature produced by change of roughness combined with changes of heat flux at the surface, and diffusion of heat or a scalar pollutant from a line source at or near ground level. The predictions are compared with observations by Rider (1952) of the flow downwind of a hedge, by Rider, Philip & Bradley (1963) of temperature and humidity downwind of a change in surface, and of vertical diffusion from a line-source at ground level.

1. Introduction

Using results established in the laboratory for boundary layers to describe the properties of the earth's boundary layer has been comparatively successful, but the usual and uncontrollable inhomogeneity of the earth's surface presents a serious obstacle to the application of results and theories established for uniform surfaces. The question of whether a surface is sufficiently uniform can be answered only if the effects of non-uniformity can be calculated, and the simplest kind of non-uniformity is a sudden change of surface roughness along a boundary at right-angles to the wind. Elliott (1958), Taylor (1962) and Panofsky & Townsend (1964) have put forward semi-empirical theories to describe the effects on the wind profile of a change of roughness, all making assumptions about the nature of the flow and requiring overall conservation of momentum without enquiring into the dynamical possibility of the flow. Both Elliott and Panofsky & Townsend assume forms for the change of flow velocity from the upstream value at the same height that are self preserving in form, i.e. the dependence on distance from the change of surface is described completely by changes of the characteristic scales of velocity and length, and which allow a logarithmic distribution of velocity very close to the surface. In a first paper (Townsend 1965, referred to as I), it was shown that self-preserving development of this kind is dynamically possible if (1) the total depth of the boundary layer is much larger than l_0, the depth of the modified layer, (ii) the ratio of the velocity change to the local velocity is small, and (iii) $\log l_0/z_0$ is large where z_0 is the roughness length. For self-preserving development, the dependence on distance of the two scales is found in terms of a non-dimensional distribution function whose form depends on the vertical transport of momentum, but any distribution which becomes logarithmic at small heights gives substantially the same results. The basic concepts are easily developed for other changes of roughness and to describe temperature fields caused by changes of roughness and changes of surface temperature.

Even in the atmosphere, the requirement that $\log l_0/z_0$ should be large is not satisfied strongly and practical use of the results needs a higher-order approximation. In this second paper, better approximations are found by assuming the distributions of velocity and temperature to have the forms appropriate for very large $\log l_0/z_0$ and then using integral conditions—overall conservation of momentum and heat—to determine the magnitude of the scales. The procedure is similar to that used for predicting the development of turbulent boundary layers (Townsend 1961). The predictions are then compared with observations by Rider (1952) of the flow downstream of a hedge, by Rider *et al.* (1963) of temperature and humidity after flow from a tarmac surface to irrigated grass and of diffusion from line-sources near ground level. Effects of thermal instability are also discussed.

The notation and co-ordinates are those used in I, i.e.

Ox is in the wind direction,
z_1 is the roughness length for $x < 0$,
z_0 is the roughness length at $x > 0$,
$M = \log z_1/z_0$,
$U_1(z)$ is the mean velocity at height z for $x < 0$,
$U(x,z)$ is the mean velocity at (x,z),
$\delta(x,z)$ is the net displacement of the streamline through (x,z),
u_1 is the friction velocity for $x < 0$,
τ_0 is the surface stress for $x > 0$,
u_0, v_0 are scales of velocity,
l_0 is a scale of length,
$\eta = z/l_0$,
T is the mean temperature relative to the surface temperature for $x < 0$,
T_g is the surface temperature for $x > 0$,
Q is the local (thermometric) heat flux,
Q_0 is the surface flux for $x > 0$,
Q_1 is the surface flux for $x < 0$,
k is the Karman constant, nearly 0·41.

2. Change of surface roughness—velocity field

As in paper I, the change of mean velocity at a particular height is expressed as the sum of a flow-acceleration term and a term representing the velocity change due to displacement of the streamlines, i.e.

$$U - U_1 = -\frac{u_1 \delta}{kz} + \frac{u_0}{k} f(z/l_0). \tag{2.1}$$

From §3 of I, the velocity change due to streamline displacement is

$$-\frac{u_1\delta}{kz} = \frac{u_0}{k(\log l_0/z_1 - C_0)}\eta^{-1}\int_0^\eta f(x)\,dx, \tag{2.2}$$

where
$$C_0 = -\int_0^\infty f(\eta)\log\eta\,d\eta \Big/ \int_0^\infty f(\eta)\,d\eta.$$

When η is small, the velocity distribution is logarithmic in form, i.e.

$$U = \frac{\tau_0^{\frac{1}{2}}}{k}\log\frac{z}{z_0},$$

and
$$U_1 = \frac{u_1}{k}\log\frac{z}{z_1}.$$

For consistency with (2.1), it is necessary that

$$f(\eta) = \log\eta + C \quad \text{for small} \quad \eta,$$

that
$$\tau_0^{\frac{1}{2}} = u_1 + u_0\{1 + (\log l_0/(z_1 - C_0)^{-1}\} \tag{2.3}$$

and that
$$u_0\{\log l_0/z_0 - C + 1 + (\log l_0/z_1 - C_0)^{-1}\} + u_1 M = 0. \tag{2.4}$$

To obtain a second relation between the scales of velocity and length, we use the condition of overall conservation of momentum,

$$\frac{dP_x}{dx} = u_1^2 - \tau_0,$$

where P_x is the total change in momentum flux across the plane x = constant, and was shown in I to be

$$P_x = \frac{u_1 u_0 l_0}{k^2}(\log l_0/z_0 - C_0)\int_0^\infty f(\eta)\,d\eta + O\left(\frac{u_0^2 l_0}{k^2} \text{ or } \frac{u_1 u_0 l_0}{k^2}\log^{-1}\frac{l_0}{z_0}\right). \tag{2.5}$$

To the approximation of small $\log l_0/z_0$ which is used throughout this paper, the condition is that

$$\frac{d}{dx}\left[I_0 l_0 \frac{\log l_0/z_1 - C_0}{\log l_0/z_0 - C + 1}\right] = 2k^2 \frac{\log l_0/z_0 - C + 2 - \frac{1}{2}M}{(\log l_0/z_0 - C + 1)^2}, \tag{2.6}$$

where
$$I_0 = -\int_0^\infty f(\eta)\,d\eta.$$

To the approximation, the solution is

$$l_0(\log l_0/z_0 - 2 - \tfrac{1}{2}M - C_0 + O(\log^{-1} l_0/z_0)) = 2k^2 I_0^{-1} x. \tag{2.7}$$

So far the distribution function $f(\eta)$ has not been specified nor has l_0 been defined. The basic requirement is that the calculated flow should have the same behaviour as the asymptotic, self-preserving flow for very large $\log l_0/z_0$. Necessarily then, $I_0 = 1$ for equation (2.7) to approach the self-preserving form, and since $f(\eta) = \log\eta + C$ for small η, little variation in the form of the function is possible. Three possibilities were compared in figure 2 of I:

(i) the mixing-length profile,

$$f(\eta) = -\int_\eta^\infty \frac{e^{-x}}{x}\,dx;$$

(ii) the 'Elliott' profile,

$$f(\eta) = \log\eta \quad \text{for} \quad \eta < 1,$$
$$= 0 \quad \text{for} \quad \eta > 1,$$

and (iii) the 'Panofsky & Townsend' profile,

$$f(\eta) = \log\tfrac{1}{2}\eta + (1-\tfrac{1}{2}\eta) \quad \text{for} \quad \eta < 2,$$
$$= 0 \quad \text{for} \quad \eta > 2.$$

The constants C and C_0 are listed in table 1. For a moderately small value of $\log l_0/z_0$ of six and small M, the greatest difference between the predicted values of l_0 is about 20 %, but the difference in predicted change of friction velocity is

Profile	C	C_0	$2+C_0$
Mixing length	$\gamma = 0{\cdot}577$	$1+\gamma = 1{\cdot}577$	3·577
Elliott	0	2	4
Panofsky & Townsend	$1-\log 2 = 0{\cdot}307$	$\frac{5}{2}-\log 2 = 1{\cdot}807$	3·807

TABLE 1

only 3 % and the differences between the predicted velocity profiles are also about 3 % expressed as fractions of the local velocity. The constants in the development equation are not those given by Panofsky & Townsend (1964) for the same shape of profile. The difference arises from the inclusion of the effects of streamline displacement and avoidance of the difficulty in the original method of reconciling velocity continuity and conservation of mass. The profiles for the three choices of $f(\eta)$ are:

(i) mixing length,

$$U-U_1 = -\frac{u_0}{k}\left[\{1+(\log l_0/z_1-C_0)^{-1}\}\int_\eta^\infty \frac{e^{-x}}{x}dx + \frac{1-e^{-\eta}}{\eta(\log l_0/z_1-C_0)}\right],$$

(ii) Elliott,

$$U-U_1 = u_0/k\,[\{1+(\log l_0/z_1-C_0)^{-1}\}\log\eta - (\log l_0/z_1-C_0)^{-1}] \quad \text{for} \quad \eta < 1,$$
$$= -u_0/k\,(\log l_0/z_1-C_0)^{-1}\,\eta^{-1} \quad \text{for} \quad \eta > 1,$$

(iii) Panofsky & Townsend,

$$U-U_1 = u_0/k[\{1+(\log l_0/z_1-C_0)^{-1}\}\{\log\tfrac{1}{2}\eta+1-\tfrac{1}{2}\eta\} - (1-\tfrac{1}{4}\eta)/(\log l_0/z_1-C_0)] \quad \text{for} \quad \eta < 2,$$
$$= -u_0/k\,(\log l_0/z_1-C_0)^{-1}\,\eta^{-1} \quad \text{for} \quad \eta > 2.$$

The streamline displacements are very nearly the same for any choice of $f(\eta)$. In particular, the displacement in the unmodified flow is δ_1, given by

$$\delta_1/l_0 = \frac{-M}{(\log l_0/z_0 - C+1)\,(\log l_0/z_0 - M - C_0)}. \tag{2.8}$$

For $\log l_0/z_0 = 6$ and M not too large, $\delta_1/l_0 \approx -0{\cdot}04\,M$ and the net streamline displacement is fairly small compared with the thickness of the modified region.

3. Line roughness—flow downwind of a fence

Following § 5 of I, the velocity change is put as

$$U - U_1 = -\frac{u_1 \delta}{kz} + \frac{u_0}{k} f(z/l_0) + \frac{v_0}{k} g(\eta), \tag{3.1}$$

with an acceleration term composed of a 'wall' component $(u_0/k) f(\eta)$, and a 'wake' component $(v_0/k) g(\eta)$. For small values of η, $f(\eta) = \log \eta + C$ while $g(\eta)$ approaches one. Both are small for large values of η. For consistency of the form (2.1) with the logarithmic variation of velocity, it is necessary that

$$\tau_0^{\frac{1}{2}} = u_1 + u_0\{1 + (\log l_0/z_0 - C_0)^{-1}\}, \tag{3.2}$$

and that

$$u_0 \left[\log l_0/z_0 \{1 + (\log l_0/z_0 - C_0)^{-1}\} - C + \frac{C-1}{\log l_0/z_0 - C_0} \right] = v_0\{1 + (\log l_0/z_0 - C_0)^{-1}\},$$

or, to the usual approximation, that

$$u_0(\log l_0/z_0 - C) = v_0. \tag{3.3}$$

The additional flux of momentum is

$$P_x = (\log l_0/z_0 - C_0) \frac{u_1 l_0}{k^2} \left[u_0 \int_0^\infty f(\eta)\, d\eta + v_0 \int_0^\infty g(\eta)\, d\eta \right] + \frac{v_0^2 l_0}{k^2} \int_0^\infty (g(\eta))^2\, d\eta, \tag{3.4}$$

and overall conservation of momentum requires that

$$dP_x/dx = u_1^2 - \tau_0 = -2u_1 u_0\{1 + (\log l_0/z_0 - C_0)^{-1}\} - u_0^2. \tag{3.5}$$

To obtain definite results, substitute the special forms of the distribution functions

$$f(\eta) = -\int_\eta^\infty \frac{e^{-x}}{x} dx, \quad g(\eta) = e^{-\eta},$$

the last being the asymptotic form for mixing-length transfer. Then $C = \gamma$ and $C_0 = \gamma$ nearly, and

$$P_x = u_1 v_0 l_0 / k^2 (\log l_0/z_0 - \gamma - 1 + \tfrac{1}{2} v_0/u_1). \tag{3.6}$$

To solve the momentum equation, we assume that the development equation has the form

$$dl_0/dx\, (\log l_0/z_0 - A) = 2k^2, \tag{3.7}$$

where A is a constant. The solution is then

$$v_0 l_0 (\log l_0/z_0 - \gamma - 1 + \tfrac{1}{2} v_0/u_1) \propto (\log l_0/z_0 - \gamma - z + A)^{-1}, \tag{3.8}$$

the constant of proportionality depending on the characteristics of the line roughness. To find the constant A, a second condition must be found and the choice is not unique. We choose to require that the first moment of the fluid acceleration should equal the first moment of the stress gradient, i.e.

$$\int_0^\infty z \frac{DU}{Dt} dz = \int_0^\infty z \frac{\partial \tau}{\partial z} dz = \int_0^\infty (u_1^2 - \tau)\, dz.$$

By definition of the streamline displacement,

$$\frac{DU}{Dt} = \left\{U_1 + \frac{u_0}{k}f(\eta) + \frac{v_0}{k}g(\eta)\right\}\left(\frac{\partial}{\partial x}\left\{\frac{u_0}{k}f(\eta) + \frac{v_0}{k}g(\eta)\right\} + \frac{d\delta}{dx}\frac{\partial}{\partial z}\left(\frac{u_0}{k}f(\eta) + \frac{v_0}{k}g(\eta)\right)\right), \quad (3.9)$$

where $$\delta = -\frac{l_0}{\log l_0/z_0 - \gamma}\left\{\frac{u_0}{u_1}\int_0^\eta f(x)\,dx + \frac{v_0}{u_1}\int_0^\infty g(x)\,dx\right\}\eta^{-1}.$$

From equation (3·8), $v_0 l_0$ varies as $(\log l_0/z_0)^{-2}$ for not too large v_0/u_1, and the term involving streamline inclination is negligible. From equations (3.3) and (3.8),

$$\frac{l_0}{v_0}\frac{dv_0}{dl_0} = -1 - \frac{2}{\log l_0/z_0},$$

and $$\frac{l_0}{u_0}\frac{du_0}{dl_0} = -1 - \frac{3}{\log l_0/z_0},$$

and $$\int_0^\infty z\frac{Du}{Dt}dz = \frac{u_1 v_0 l_0}{k^2}\frac{dl_0}{dx}(\log l_0/z_0 - \tfrac{1}{2} - \gamma). \quad (3.10)$$

Using the mixing-length variations of stress,

$$\tau = 2u_1 u_0 F(\eta) + 2u_1 v_0 G(\eta),$$
$$= 2u_1(u_0 e^{-\eta} - v_0 \eta e^{-\eta}),$$

we find that $$dl_0/dx\,(\log l_0/z_0 + \tfrac{1}{2} - \gamma) = 2k^2, \quad (3.11)$$

and so $A = \gamma - \frac{1}{2} = 0{\cdot}077$.

These results do not involve an assumption of small disturbance and should give a good account of a real flow. For a solid obstacle such as a fence or hedge, the effective height, i.e. the value of l_0 extrapolated back to the position of the obstacle, should be near h_0 defined by

$$F_0 = C_d U_1^2(h_0)\,h_0 = C_d \frac{u_1^2 h_0}{k^2}\log^2 h_0/z_0, \quad (3.12)$$

where F_0 is the force per unit length exerted by the obstacle. The constant of proportionality in (3.8) should then be chosen so that $-P_x = F_0$ when $l_0 = h_0$. A convenient form of the final result is

$$\frac{\tau_0}{u_1^2} = 1 - \frac{h_0(\log h_0/z_0 - \gamma - 1)^2(\log h_0/l_0 - \frac{5}{2})}{l_0(\log l_0/z_0 - \gamma - 1)^2(\log l_0/z_0 - \frac{5}{2})}, \quad (3.13)$$

for an obstacle of effective height h_0. If the line roughness adds momentum to the flow, the sign before the quotient is positive.

Rider (1952) has published measurements of wind velocities downwind of a hedge of height 1·6 m, and his observations at heights of 0·5 and 1 m can be compared with the predictions of the theory. The full profile for our choice of distribution function is

$$U - U_1 = \frac{v_0}{k}\left\{e^{-\eta} - (\log l_0/z_0 - \gamma - 1)^{-1}\int_\eta^\infty \frac{e^{-x}}{x}dx + (\log l_0/z_0 - \gamma + 1)^{-1}\frac{1 - e^{-\eta}}{\eta}\right\}, \quad (3.14)$$

and, for small η,

$$U - U_1 = \frac{v_0}{k}\left\{e^{-\eta} - 1 + \frac{\log z/z_0 + \eta}{\log l_0/z_0 - \gamma - 1}\right\}. \quad (3.15)$$

Then,
$$\frac{U-U_1}{U_1} = \frac{\tau_0^{\frac{1}{2}}-u_1}{u_1}\left\{1+\frac{\log l_0/z_0-\gamma-2}{\log z/z_0}(e^{-\eta}-1)\right\} \tag{3.16}$$

to a fair approximation, where τ_0 is given by equation (3.14) and l_0 by equation (3.11) with the condition that l_0 is of order the hedge height at the position of the

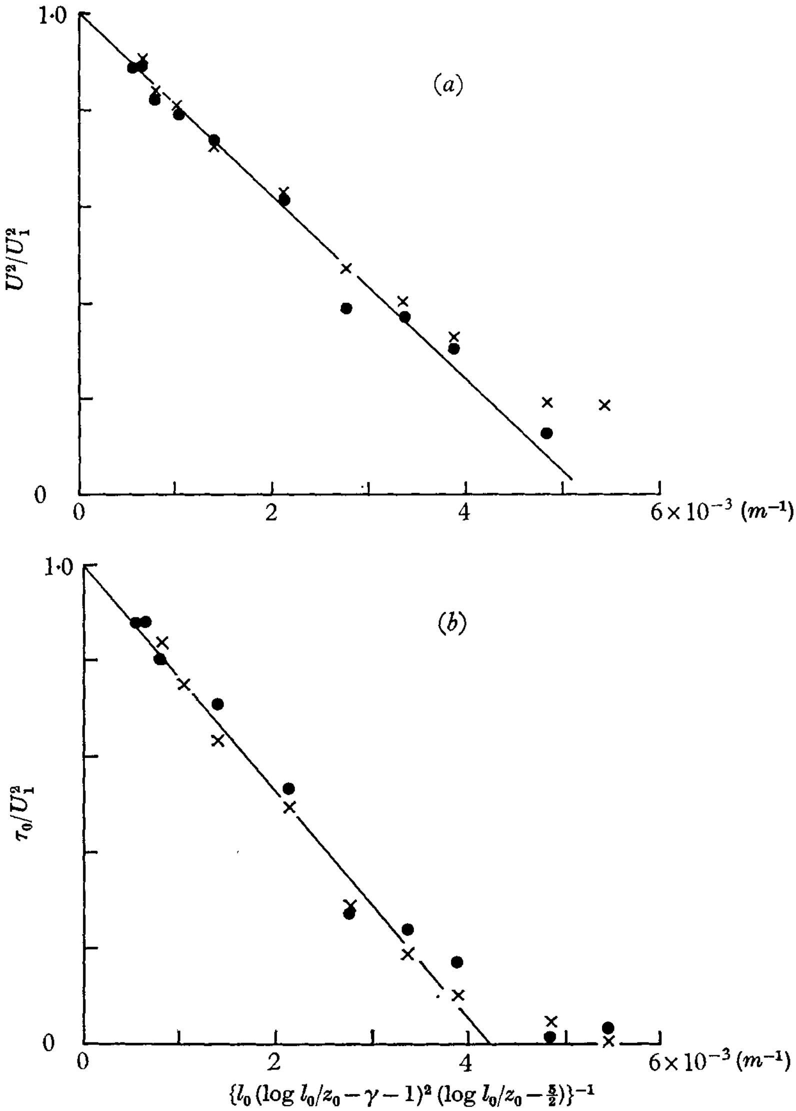

FIGURE 1. Comparison of observations by Rider (1952) downwind of a hedge with theory. τ_0/u_1^2 has been calculated from the observed values of U^2/U_1^2, using equation (3.16). ●, Observations at 0·5 m; ×, observations at 1 m.

hedge. The results of comparing the observations with the theory are shown in figure 1, first omitting and then including the variation of U/U_1 with height as given by (3.16). The roughness length z_0 was calculated from the observed profiles, and the fit depends on the choice of three parameters. The first is u_1, the value of the friction velocity far downstream of the hedge. Using friction velocities computed from the winds at the lowest height of observation, equation (3.13) indicates a value of u_1 of 1·06 times the value at 107 m from the hedge, for any reasonable choice of the other two parameters. The second is really the constant of integration for equation (3.11), fixing the effective zero of horizontal distance, but it is convenient to use the effective value of l_0 at the hedge position. The third, h_0, the value of l_0 for which the surface stress extrapolated by equation (3.13) is zero, is determined by the slope of the line drawn in figure 1 (*a*). The selected value for l_0 at the hedge is 60 cm and then the observations are consistent with $h_0 = 1{\cdot}23$ m. The last two parameters are not entirely independent for they should both be of the same order of magnitude as the height of the hedge, and the drag coefficient C_d should be of order one. In fact, the calculated value of C_d is 0·9. Agreement with the theory is good for measurements taken more than 15 m from the hedge.

4. Temperature variations caused by change of surface

If the density changes are too small to affect the motion, the temperature field in a flow is linearly dependent on the thermal boundary conditions. The solution of the change-of-roughness flow with constant but different heat flux from the surface on the two sides of the function can be obtained by superposition of the solutions for two special cases, (i) zero flux upstream, $Q_1 = 0$, and (ii) $Q_1 = Q_0$, constant flux on both sides.

With zero flux upwind of the change of surface, the temperature for negative x is independent of position and is taken as the zero level of temperature. The self-preserving distribution of temperature for positive x is

$$T = -(\theta_0/k)\,\phi_1(z/l_0) \tag{4.1}$$

and must coincide with the logarithmic distribution

$$T = T_g - (Q_0/k\tau_0^{\frac{1}{2}})\log z/z_0 \tag{4.2}$$

when η is small. T_g is the surface temperature and it has been assumed that the diffusivities for heat and momentum are equal in an equilibrium layer. For consistency,

$$\phi_1(\eta) = \log\eta + C_1 \quad \text{for small} \quad \eta,$$

$$\theta_0 = Q_0/\tau_0^{\frac{1}{2}}, \tag{4.3}$$

$$T_0 = \theta_0/k(\log l_0/z_0 - C_1). \tag{4.4}$$

Overall conservation of heat requires that the additional heat flux in the layer Q_x should be equal to $Q_0 x$. In I, it was shown that

$$Q_x = -\frac{u_1\theta_0 l_0}{k^2}\left\{\log l_0/z_1\int_0^\infty \phi_1(\eta)\,d\eta + \int_0^\infty \phi_1(\eta)\log\eta\,d\eta\right\}, \tag{4.5}$$

and, if an asymptotic form for the function $\phi_1(\eta)$ is used, $\int_0^\infty \phi_1(\eta)\,d\eta = -\frac{1}{2}$ by equation (7·11) of I and so

$$l_0\left(\log l_0/z_0 - M - 2\int_0^\infty \phi_1(\eta)\log\eta\,d\eta\right) = 2k^2\frac{\tau_0^{\frac{1}{2}}}{u_1}x. \tag{4.6}$$

From equations (2.3) and (2.4),

$$\tau_0^{\frac{1}{2}}/u_1 = 1 - M(\log l_0/z_0 - C + 1)^{-1} + O(\log l_0/z_0)^{-2},$$

and to the approximation in use,

$$l_0\left(\log l_0/z_0 - 2\int_0^\infty \phi_1(\eta)\log\eta\,d\eta\right) = 2k^2x. \tag{4.7}$$

Corresponding to the three distribution functions for the change-of-roughness flow are three forms of $\phi_1(\eta)$ which lead to the following results:

(i) Mixing-length profile:

$$\left.\begin{aligned}
&\phi_1(\eta) = -\int_\eta^\infty \frac{e^{-2x}}{x}\,dx,\\
&\theta_0 = \frac{Q_0}{u_1}\left(1-\frac{M}{\log l_0/z_0-\gamma+1}\right)^{-1},\\
&l_0(\log l_0/z_0 - 1-\gamma-\log 2) = 2k^2x,\\
&T_g = Q_0/ku_1\,(\log l_0/z_0-\gamma-\log 2+M),
\end{aligned}\right\} \tag{4.8}$$

(ii) Elliott profile:

$$\left.\begin{aligned}
&\phi_1(\eta) = \log 2\eta \quad \text{for} \quad \eta < \tfrac{1}{2},\\
&\qquad\;\; = 0 \qquad\quad \text{for} \quad \eta > \tfrac{1}{2},\\
&\theta_0 = \frac{Q_0}{u_1}\left(1-\frac{M}{\log l_0/z_0+1}\right)^{-1},\\
&l_0(\log l_0/z_0 - 2-\log 2) = 2k^2x,\\
&T_g = Q_0/ku_1\,(\log l_0/z_0-\log 2+M),
\end{aligned}\right\} \tag{4.9}$$

(iii) Panofsky & Townsend profile:

$$\left.\begin{aligned}
&\phi_1(\eta) = \log\eta + (1-\eta) \quad \text{for} \quad \eta < 2,\\
&\qquad\;\; = 0 \qquad\qquad\quad\; \text{for} \quad \eta > 2,\\
&\theta_0 = \frac{Q_0}{u_1}\left(1-\frac{M}{\log l_0/z_0-\log 2}\right)^{-1},\\
&l_0(\log l_0/z_0 - \tfrac{5}{2}) = 2k^2x,\\
&T_g = Q_0/ku_1\,(\log l_0/z_0-1+M).
\end{aligned}\right\} \tag{4.10}$$

The greatest divergence of predicted temperature is at the surface, with a difference of $\gamma Q_0/(ku_1)$ between the first two.

The self-preserving distribution for the constant flux situation $Q_1 = Q_0$ is

$$T - T_1 = -\frac{\theta_0}{k}\phi_2(z/l_0) + \frac{Q_1\delta}{ku_1 z}, \tag{4.11}$$

which includes a term representing the change due to streamline displacement. Consistency with the logarithmic distribution requires that

$$\left.\begin{aligned}
&\phi_2(\eta) = \log\eta + C_2 \quad \text{for small} \quad \eta,\\
\text{that}\qquad &\theta_0 = \frac{Q_0}{u_1}\left\{\frac{\tau_0^{\frac{1}{2}}-u_1}{\tau_0^{\frac{1}{2}}} + \frac{u_0}{u_1(\log l_0/z_1 - C_0)}\right\},
\end{aligned}\right\} \tag{4.12}$$

$$ku_1T_g = MQ_0 - Q_0\left\{\frac{\tau_0^{\frac{1}{2}}-u_1}{\tau_0^{\frac{1}{2}}}(\log l_0/z_0 - C_2) - \frac{u_0}{u_1}\frac{C_2-C+1}{\log l_0/z_1 - C_0}\right\}. \tag{4.13}$$

Overall conservation of heat requires that the additional heat flux should be zero, i.e.

$$Q_x = -\frac{\theta_0 u_1 l_0}{k^2}\left\{\log\frac{l_0}{z_1}\int_0^\infty \phi_2(\eta)\,d\eta + \int_0^\infty \phi_2(\eta)\log\eta\,d\eta\right\} = 0. \tag{4.14}$$

If a form for $\phi_2(\eta)$ valid for large values of $\log l_0/z_0$ is used, $\int_0^\infty \phi_2(\eta)\,d\eta = 0$, and the condition cannot be satisfied with a reasonable choice of $\phi_2(\eta)$. In I, the asymptotic profile obtained by assuming two components of mixing-length transfer one connected with the changes in eddy diffusivity due to the changes in Reynolds stress and the other connected with variations of heat flux in the modified region. The origin of the two components suggests that the length scale of the first component should be the same as the length scale of the velocity changes and given by the theory of § 2, while the scale of the other component varies so as to allow overall conservation of heat. The mixing-length profile of I (equation (7.22)) can be changed in this way to

$$\phi_2(\eta) = \int_\eta^\infty \frac{e^{-x}}{x}\,dx - 2\int_\eta^\infty \frac{e^{-2\alpha x}}{x}\,dx, \tag{4.15}$$

the first term being connected with change of eddy diffusivity and expected to behave in the same way as the velocity-distribution function. The second term has a scale whose ratio to that of the first varies slowly, i.e. α is a function of $\log l_0/z_0$. Substituting in the equation for heat conservation, we find that

$$(1-1/\alpha)(\log l_0/z_0 - M - 1 - \gamma - \log 2) + \log 2\alpha = 0. \tag{4.16}$$

To our approximation, $C_2 = \log 2 + \gamma$, and

$$\theta_0 = -\frac{MQ_0}{u_1}\left\{\frac{1}{\log l_0/z_0 - \gamma + 1} + \frac{M - \log 2}{(\log l_0/z_0 - \gamma + 1)^2}\right\}, \tag{4.17}$$

$$k u_1 T_g = MQ_0\left\{2 + \frac{M - \log 2}{\log l_0/z_0 - \gamma + 1}\right\}, \tag{4.18}$$

with l_0 given by

$$l_0(\log l_0/z_0 - \tfrac{1}{2}M - 3 - \gamma) = 2k^2 x. \tag{4.19}$$

No equivalent of the Elliott profile exists with the necessary properties, but the Panofsky & Townsend profile can be modified in an analogous way to

$$\left.\begin{aligned}\phi_2(\eta) &= \log\tfrac{1}{2}\eta + \alpha(1-\tfrac{1}{2}\eta) \quad &&\text{for}\quad \eta < 2,\\ &= 0 &&\text{for}\quad \eta > 2.\end{aligned}\right\} \tag{4.20}$$

The condition (4.14) leads to

$$(\alpha - 2)(\log l_0/z_0 - \tfrac{3}{2} + \log 2 - M) + 1 = 0, \tag{4.21}$$

and $C_2 = 2 - \log 2$ very nearly. Then

$$\theta_0 = -\frac{MQ_0}{u_1}\left\{\frac{1}{\log l_0/z_0 + \log 2} + \frac{M-1}{(\log l_0/z_0 + \log 2)^2}\right\}, \tag{4.22}$$

$$k u_1 T_g = MQ_0\left\{2 + \frac{M-1}{\log l_0/z_0 + \log 2}\right\}, \tag{4.23}$$

with l_0 given by

$$l_0(\log l_0/z_0 - \tfrac{1}{2}M - \tfrac{9}{2} + \log 2) = 2k^2x. \tag{4.24}$$

Although the Elliott kind of profile does not satisfy the basic condition (4.14), it is still a fair approximation to the distributions which do when used with the form (4.9) and length scales given by either of the more self-consistent profiles. The temperature scale and surface temperature are

$$\theta_0 = -\frac{MQ_0}{u_1}\left\{\frac{1}{\log l_0/z_0+1} + \frac{M-\log 2}{(\log l_0/z_0+1)^2}\right\}, \tag{4.25}$$

and

$$ku_1T_g = MQ_0\left\{2 + \frac{M-\log 2}{\log l_0/z_0+1}\right\}. \tag{4.26}$$

5. Effects of thermal instability

So far the buoyancy forces have been assumed to be too small to affect the turbulent motion in the region of modified flow, and the condition for this to be true is that the depth of the region should be small compared with the Monin–Obukhov length (Priestley 1959) $L = \tau_0^{\frac{3}{2}}T_m/(kgQ)$ (T_m is the absolute temperature of the flow). If the ratio is not small, the existence of two distinct scales of length means that self-preserving development cannot occur except in very special circumstances, e.g. with characteristic L proportional to l_0. Even though self-preserving flow is no longer a possibility, it remains true that most of the variation of velocity and temperature takes place in the new equilibrium layer which has about one fifth the thickness of the whole modified layer, and a fair approximation to the distributions can be obtained by assuming the equilibrium profiles to extend to the extreme edge of the modified region. In a diabatic, constant-stress, equilibrium layer, dimensional reasoning leads to the relation,

$$\frac{dU}{dz} = \frac{\tau_0^{\frac{1}{2}}}{kz}f_n(z/L) \tag{5.1}$$

in which the function approaches one for small values of z/L, and approaches zero for large z/L, probably as $(z/L)^{-n}$ where n may be about 4/3. For fairly small z/L,

$$\frac{dU}{dz} = \frac{\tau_0^{\frac{1}{2}}}{kz}(1-\alpha z/L)$$

where $\alpha \approx 0{\cdot}6$ (Priestley 1959), and

$$U = \frac{\tau_0^{\frac{1}{2}}}{k}\left\{\log\frac{z}{z_0} - \alpha\frac{z}{L}\right\}. \tag{5.2}$$

For very large z/L, since velocity gradients become very small,

$$U = \tau_0^{\frac{1}{2}}/k(\log L/z_0 + A'), \tag{5.3}$$

where A' is possibly about 0·4. To apply these results to the modified layer, a representative value of the length L must be chosen and, if stress and heat flux are functions of height, it should be based on the stress and flux near the middle of the modified region. Two special cases will be considered, (*a*) unchanged ground flux of sensible heat, and (*b*) zero flux downwind of the change of roughness.

For unchanged flux and small departures from neutral conditions, the results for self-preserving development lead to

$$\frac{\tau^{\frac{3}{2}}T_m}{kgQ} = L_1\left[1+\frac{u_0}{u_1}\{3\,F(\eta)+\phi_2(\eta)\}\right], \tag{5.4}$$

where $F(\eta)$ and $\phi_2(\eta)$ are distribution functions defined in I, §§4 and 7. From the forms of F and ϕ_2, it appears likely that typical values of $3F+\phi_2$ will not be far from $+1$, and so we assume an effective value of the Monin–Obukhov length

$$L = L_1(1+Bu_0/u_1), \tag{5.5}$$

where L_1 is the upwind value, and $B \approx 1$. Then the velocity distribution at the point considered is

$$U = \frac{u_1+u_0}{k}\left(\log\frac{z}{z_0}-\alpha\frac{z}{L}\right),$$

compared with

$$U_1 = \frac{u_1}{k}\left(\log\frac{z}{z_1}-\alpha\frac{z}{L_1}\right)$$

upwind. A condition of no flow acceleration beyond $z = l_0$ would require that

$$U_1(l_0) = U(l_0)+u_1\delta_1/kl_0\,(1-\alpha l_0/L_1),$$

but the displacement term is small and can be neglected if the main purpose is to estimate the influence of static instability. Then, for small l_0/L_1,

$$\frac{u_0}{u_1} = -\frac{M}{\log l_0/z_0-\alpha(1-B)\,l_0/L_1}. \tag{5.6}$$

For large l_0/L_1,

$$U_1 = \frac{u_1}{k}(\log L_1/z_1+A'),$$

$$U = \frac{u_1+u_0}{k}(\log L/z_0+A'),$$

and

$$\frac{u_0}{u_1} = -\frac{M}{\log L_1/z_0+A'}. \tag{5.7}$$

With zero ground flux downstream of the change of surface, the effective value of L is a nearly constant multiple of L_1, since $\tau^{\frac{3}{2}}T_m/(kgQ)$ decreases from an infinite value at the surface to L_1 in the unmodified flow. For small l_0/l_1, the condition $U_1(l_0) = U(l_0)$ leads to

$$\frac{u_0}{u_1} = -\frac{M+\alpha l_0\,(L_1^{-1}-L^{-1})}{\log l_0/z_0-\alpha l_0/L}. \tag{5.8}$$

For large l_0/L_1,

$$\frac{u_0}{u_1} = -\frac{M+\log L/L_1}{\log L/z_0+A'}. \tag{5.9}$$

The variation of u_0/u_1 with l_0/L_1 is indicated in figure 2 for several values of M.

The results obtained by matching an equilibrium distribution to the undisturbed flow are not complete without a knowledge of the dependence of l_0 on fetch. In the earlier sections, the problem was approached by using the condition of overall conservation of momentum to derive an equation for the length scale, but there is now no reason to suppose that the excess momentum flux calculated

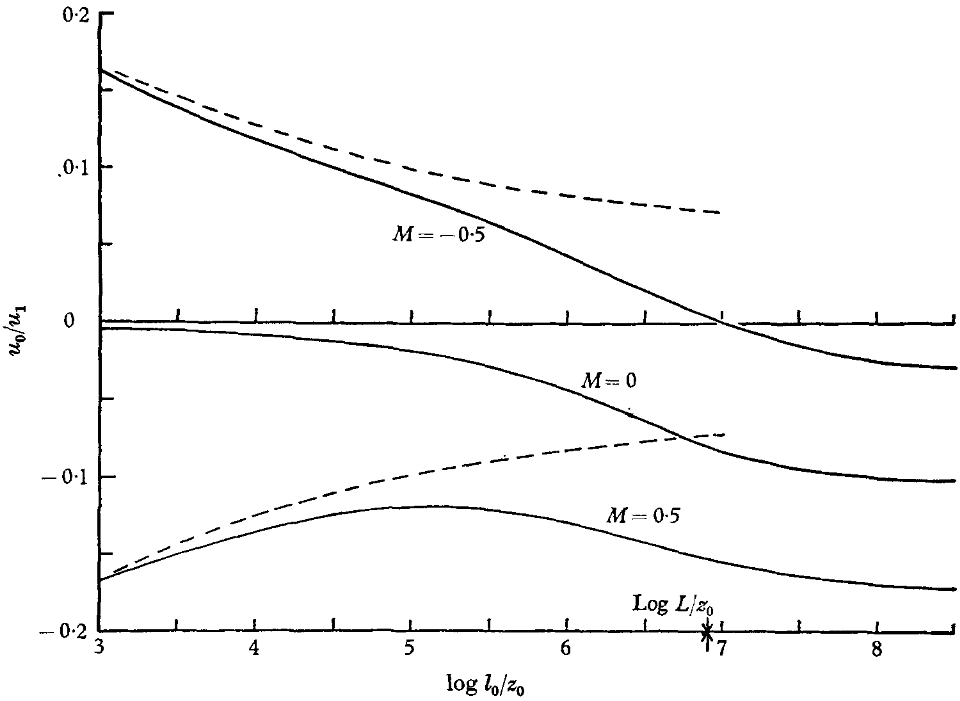

FIGURE 2. Dependence of change of surface friction on log l_0/z_0 for unstable conditions of kind (*b*) from equations (5.8) and (5.9). It is assumed that $L = 2L_1 = 10^3 z_0$, $\alpha = 0{\cdot}6$ and $A' = 0$. The broken lines show the behaviour for neutral stability.

from the equilibrium profile behaves in a similar way to the excess flux in the real flow. In fact, assumption that it does leads to most implausible results for moderate values of l_0/L_1. A similar situation occurs in the theory of temperature changes after a change of roughness with constant surface-heat-flux (§ 4). There an equilibrium temperature distribution is quite incapable of satisfying conservation of heat, and appeal was made to the physical causes of the temperature changes for a guide to the magnitude of the length scale. Here the extension of the modified region into the undisturbed flow may be regarded as a diffusion of momentum from the ground with an effective diffusivity determined by the turbulent motion at heights of order l_0. This model, which is related to ones used by Cermak (1963) and by Miyake (see Panofsky & Townsend 1964), leads to

$$U(l_0)\, dl_0^2/dx = 4\nu_T(l_0), \tag{5.10}$$

where the factor 4 permits the equation to represent the behaviour of l_0 for adiabatic conditions, and $\nu_T(l_0)$ is the eddy diffusivity for momentum at height l_0. For the upwind profile and small l_0/L_1,

$$\nu_T = ku_1 z(1-\alpha z/L_1)^{-1},$$

and so
$$\frac{dl_0}{dx}(\log l_0/z_1-\alpha l_0/L_1)(1-\alpha l_0/L) = 2k^2\frac{u_1+u_0}{u_1}, \qquad (5.11)$$

reducing to the asymptotic form for small l_0/L_1 and large $\log l_0/z_0$. For either case considered, the equation integrates to

$$l_0(1-\tfrac{1}{2}\alpha l_0/L)(\log l_0/z_0-1) = 2k^2 x, \qquad (5.12)$$

showing that l_0 is increased by instability in the ratio $(1-\frac{1}{2}\alpha l_0/L)^{-1}$. Substituting the changed value of l_0, the following results for change in surface friction are found:

(a) unchanged flux ($Q_1 = Q_0$),

$$\frac{u_0}{u_1} = \left(\frac{u_0}{u_1}\right)_{\text{neutral}} \times \left\{1+\frac{\alpha(B-\frac{1}{2})\,l_0/L_1}{\log l_0/z_0}\right\}^{-1}, \qquad (5.13)$$

(b) zero flux downwind ($Q_0 = 0$),

$$\frac{u_0}{u_1} = \left(\frac{u_0}{u_1}\right)_{\text{neutral}} \times \left\{1+\frac{\alpha l_0}{M}\left(\frac{1}{L_1}-\frac{1}{L}\right)-\frac{1}{2}\frac{\alpha l_0/L}{\log l_0/z_0}\right\}, \qquad (5.14)$$

both for small values of l_0/L_1.

6. Change of surface—comparison with observations

Panofsky & Townsend (1964) have analysed observations of wind profiles downstream of a change of roughness and find them to be in fair agreement both with Elliott's (1958) and their theory of the effect. The same observations are also in agreement with the mixing-length profile or indeed any plausible profile which becomes logarithmic near the surface. Use of the concept of self-preserving development of changes in flow parameters leads to predictions of temperature changes caused by a sudden change in surface roughness and surface temperature. Rider *et al.* (1963) have made careful and comprehensive observations of temperature and humidity in a boundary layer passing from a tarmac surface to well-irrigated mown grass. Over the impermeable tarmac, the upwards flux of total heat was almost entirely a flux of sensible heat, but evaporation from the grass was more than sufficient to carry the net flux of total heat and the flux of sensible heat changed sign near the ground. They compared their results with a solution by Philip (1959) of the corresponding diffusion problem which assumes no change in velocity profile, but the change of roughness was large, from $z_1 = 2\times 10^{-3}$ cm to $z_0 = 0{\cdot}14$ cm, and produced a considerable change of the velocity profile. The observations will be compared with a particular form of the theory, chosen as the simplest form that conforms to the basic requirements, and that can make allowance for the strong instability of the flow over the tarmac.

Instability of the flow affects both the velocity profiles and the temperature profiles, but the effect on the distribution of surface stress is small if the considerations of § 5 are any guide. There it was estimated that, for zero heat flux downwind of the change of surface, the change in surface stress is increased by a factor of

$$1+\alpha\frac{l_0}{L_1}\left\{M^{-1}\left(1-\frac{L_1}{L}\right)-\frac{1}{2}\frac{L_1/L}{\log l_0/z_0}\right\},$$

compared with conditions of neutral stability. Typical values of L_1 are between 1·5 and 5 m, and $M = -4{\cdot}2$ with l_0 1 m or less. The probable value of the factor is near 0·93 and the uncertainty in local friction velocity from this cause is no more than 4 %. On the other hand, the static instability over the tarmac causes the temperature profiles to depart from the neutral logarithmic form, and measured values must be used.

First, consider the flux and distribution of total heat, conveniently in the form of equivalent temperature and the corresponding thermometric flux. For these purposes, the equivalent temperature is

$$T_a = T+\frac{\rho_w L_w}{\rho_a c_p}, \tag{6.1}$$

where L_w is the latent heat of water vapour, and ρ_w/ρ_a is the mass fraction of water vapour in the air. It is nearly the temperature that the air would attain if the latent heat of the contained water vapour were released by condensation. Downstream of the change of surface where the sensible heat flux is mostly small, we assume a logarithmic distribution of equivalent temperature,

$$T_a = T_a(0)-\frac{Q_a}{k(u_1+u_0)}\log z/z_0 \quad \text{for} \quad z < \tfrac{1}{2}l_0, \tag{6.2}$$

where Q_a is the ground flux of total heat and $T_a(0)$ is the equivalent temperature at the surface. The height l_0 is calculated from

$$l_0(\log l_0/z_0-3-\log 2) = 2k^2x, \tag{6.3}$$

an expression which may be regarded as an average of the separate but similar expressions for constant-flux and step-flux conditions. Continuity of temperature at height $\frac{1}{2}l_0$ (neglecting the effects of streamline displacement) requires that

$$T_a(z) = T'_a(\tfrac{1}{2}l_0)-\frac{Q_a}{k(u_1+u_0)}\log 2z/l_0, \tag{6.4}$$

and

$$T_a(0) = T'_a(\tfrac{1}{2}l_0)+\frac{Q_a}{k(u_1+u_0)}\log(\tfrac{1}{2}l_0/z_0), \tag{6.5}$$

where $T'_a(z)$ is the equivalent temperature over the tarmac at height z. The scale temperature $Q_a/\{k(u_1+u_0)\}$ is calculated by equating Q_a, the convective flux of total heat, to the observed net radiation, i.e. neglecting the heat flux in the ground, and finding the friction velocity u_1+u_0 from the measured velocity profiles and the value of u_0/u_1 calculated from the measured roughnesses. The tabulated values are means of measurements at the edge of the tarmac and at 16 m, and so u_1 was calculated from

$$u_1 = k\frac{U_{64}-U_{27\cdot5}}{\log 64/27{\cdot}5}(1+\tfrac{1}{2}u_0/u_1)^{-1}$$

with the calculated value of u_0/u_1. (Subscripts refer to height in cm.) $T'_a(\frac{1}{2}l_0)$ was found by interpolation of the tarmac profile. To reduce numerical work and possible scatter from finite times of averaging, the observations were grouped into sets of five consecutive runs with roughly similar values of net radiation. A com-

	$x = 1$ m	$x = 4$ m		$x = 16$ m		
	$z = 5$ cm	$z = 5$ cm	$z = 11{\cdot}5$	$z = 5$	$z = 11{\cdot}5$	$z = 27{\cdot}5$
Mean error	−0·36	+0·02	−0·12	−0·14	+0·40	+0·36
(Variance)$^{\frac{1}{2}}$	0·32	0·41	0·20	0·58	0·41	0·40
Mean change	1·6	2·4	1·4	4·7	2·6	1·5
Predicted change / Observed change	0·78	1·01	0·91	0·97	1·15	1·24

TABLE 2. Analysis of prediction errors for equivalent temperature

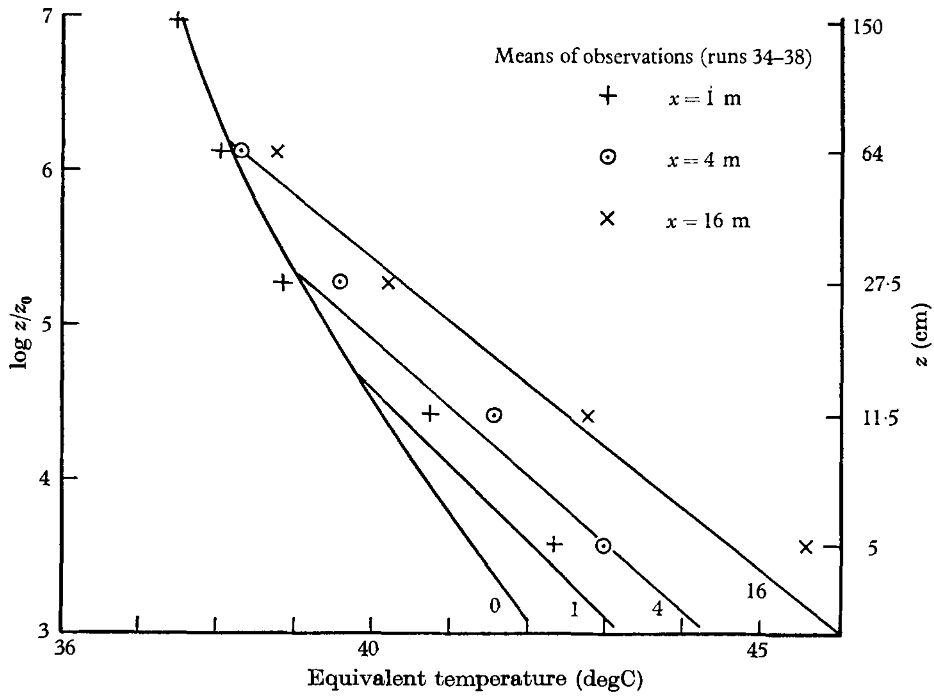

FIGURE 3. Comparison of predicted and observed values of equivalent temperature, using runs 34–38 of Rider *et al.* (1963). Numbers on the curves refer to distances in metres from the change of surface.

parison of the predicted and observed values of equivalent temperature is shown graphically in figure 3 for a set selected at random, and table 2 analyses the differences at positions within the region of appreciable change. The systematic deviations near the interface are expected from the crude angularity of the assumed profile, but predictions at lower levels are very satisfactory.

The partition of the total heat flux between sensible and latent heat depends on conditions at the ground. With continued irrigation, the likely condition is saturation at 'ground level', assumed to be the level with extrapolated zero velocity, i.e. at $z = z_0$. Extrapolating the equivalent temperature to $z = z_0$ gives $T_a(0)$ defined by (6.5), and the temperature and humidity at ground level can be found from a table of equivalent temperature of saturated air as a function of sensible temperature. It only remains to make plausible and consistent assumptions about the profiles of humidity and sensible temperature analogous with (6.2). The thermometric flux of humidity changes within the modified layer from its ground value $Q_w(0)$ to zero at height $\frac{1}{2}l_0$, and the analogous profile is

$$T_w = T_w(0) - \frac{Q_w(0)}{k(u_1+u_0)}\left(\log\frac{z}{z_0} - 2\frac{z}{l_0}\right), \tag{6.6}$$

chosen so that the humidity gradient becomes zero at $z = \frac{1}{2}l_0$. Humidity is measured here by

$$T_w = T_a - T = \frac{\rho_w L_w}{\rho_a c_p}, \tag{6.7}$$

and, at ground-level,

$$Q(0) + Q_w(0) = Q_a(0). \tag{6.8}$$

Continuity of T_w at height $\frac{1}{2}l_0$ requires that

$$\frac{Q_w(0)}{k(u_1+u_0)}\left(\log\frac{l_0}{2z_0} - 1\right) = T_w(0) - T'_w(\tfrac{1}{2}l_0), \tag{6.9}$$

and the ground fluxes and distributions are now determined.

The agreement between these predictions and the observations is analysed in tables 3 and 4, and figures 4 and 5 compare prediction and observation for the set of observations of figure 3. The predictions of temperature are the least satisfactory, but this is expected since the distribution depends on the difference between the fluxes of total heat and latent heat, both large quantities. It is clear that more elaborate and realistic distributions would improve the agreement, particularly near $z = \frac{1}{2}l_0$, but the prediction errors are comparable with effects from uncertainty in the boundary conditions. In particular, there is internal evidence that the effective roughness length of the grass changed from one day to another by anything up to 50 %. The basic steps of the calculation are:

(i) Compute u_0/u_1 from

$$2k^2x/z_0 = (-Mu_1/u_0 - 4 - \tfrac{1}{2}M)\exp(-Mu_1/u_0)$$

using the observed values of the roughness lengths.

(ii) Compute $\log(\frac{1}{2}l_0/z_0)$ from

$$k^2x/z_0 = (\log\tfrac{1}{2}l_0/z_0 - 3)\,\tfrac{1}{2}l_0/z_0.$$

(iii) Equate $\rho c_p Q_a$, the convective flux of total heat, to the measured net radiation, obtain the friction velocity u_1 from the upstream velocity profile and calculate equivalent temperatures from

$$T_a = T'_a(\tfrac{1}{2}l_0) - \frac{Q_a}{k(u_1+u_0)}\log(2z/l_0) \quad \text{for} \quad (z < \tfrac{1}{2}l_0).$$

(iv) From the ground values of the equivalent temperature

$$T_a(0) = T'_a(\tfrac{1}{2}l_0) + \frac{Q_a}{k(u_1+u_0)}\log(\tfrac{1}{2}l_0/z_0)$$

find the corresponding values of ground temperature and humidity for saturated air.

(v) Use
$$Q_w = \frac{k(u_1+u_0)}{\log \frac{1}{2}l_0/z_0 - 1}\{T_w(0) - T'_w(\tfrac{1}{2}l_0)\}$$
to find the thermometric flux of latent heat.

	x = 1 m	x = 4 m		x = 16 m		
	z = 5 cm	z = 5	z = 11·5	z = 5	z = 11·5	z = 27·5
Mean error	+0·52	+0·84	−0·70	−0·01	−0·80	−1·04
(Variance)$^{\frac{1}{2}}$	0·50	0·36	0·16	0·74	0·44	0·29
Mean change	2·3	4·9	2·85	9·3	5·7	2·8
Predicted change / Observed change	1·23	1·23	0·75	1·00	0·86	0·63

N.B. Humidity is measured as difference of equivalent and sensible temperature.

TABLE 3. Analysis of prediction errors for humidity

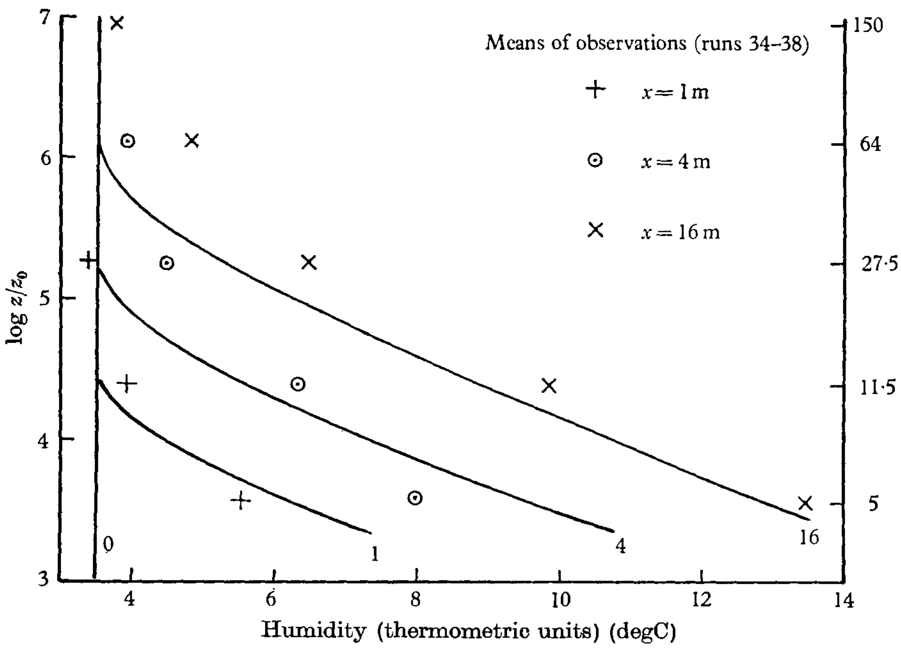

FIGURE 4. Comparison of predicted and observed values of humidity, using runs 34–38 of Rider *et al.* (1963). Numbers on the curves refer to distances in metres from the change of surface.

(vi) Calculate humidities from
$$T_w = T'_w(\tfrac{1}{2}l_0) - \frac{Q_w}{k(u_1+u_0)}\left(\log 2\frac{z}{l_0} + 1 - \tfrac{1}{2}\frac{z}{l_0}\right)$$
with $\rho_w/\rho_a = c_p T_w/L_w$.

(vii) Calculate temperatures from
$$T = T_a - T_w.$$

7. Diffusion from a line-source—comparison with observation

Diffusion from a line-source of heat is a problem closely related to the distribution caused by a step change in surface flux. In §4, it was shown that, with no change of roughness and for $Q_1 = 0$,

$$T = -(\theta_0/k)\,\phi_1(z/l_0),$$

	$x = 1$ m	$x = 4$ m		$x = 16$ m		
	$z = 5$ cm	$z = 5$	$z = 11{\cdot}5$	$z = 5$	$z = 11{\cdot}5$	$z = 27{\cdot}5$
Mean error	−0·08	−0·82	+0·58	−0·13	+0·76	+1·40
(Variance)$^{\frac{1}{2}}$	0·51	0·70	0·42	0·85	0·90	0·62
Mean change	0·7	2·5	1·45	4·6	3·1	1·3
Predicted change / Observed change	2·26	1·33	0·60	1·03	0·76	−ve

TABLE 4. Analysis of prediction errors for sensible temperature

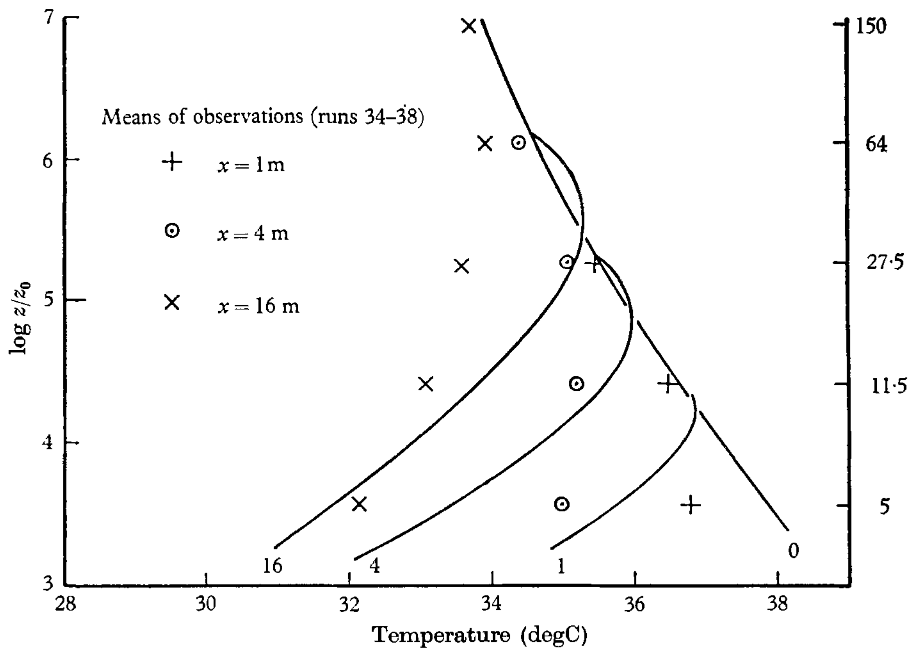

FIGURE 5. Comparison of predicted and observed values of temperature, using runs 34–38 of Rider *et al.* (1963). Numbers on curves refer to distances in metres from the change of surface.

where $\theta_0 = Q_0/u_1$ and

$$l_0\left\{\log l_0/z_0 - 2\int_0^\infty \phi_1(\eta)\log\eta\, d\eta\right\} = 2k^2x \tag{7.1}$$

to the usual approximation. To obtain the standard asymptotic form for dl_0/dx at large values of $\log l_0/z_0$, it is necessary that

$$\int^\infty \phi_1(\eta)\, d\eta = -\tfrac{1}{2}. \tag{7.2}$$

A second requirement is that, if the diffusivities for heat and momentum in an equilibrium layer are everywhere in the ratio K_h/K_m and not equal,

$$\phi_1(\eta) = (K_m/K_h)(\log\eta + C_1)$$

for small values of η. From this result, the distribution from a line-source of strength $Q_0\Delta$ is obtained by superimposing the distribution for (i) $Q(0) = 0$ for $x < 0$, $Q(0) = Q_0$ for $x > 0$ on the distribution for (ii) $Q(0) = 0$ for $x < \Delta$, $Q(0) = -Q_0$ for $x > \Delta$, where Δ is small. The composite distribution is

$$\begin{aligned} T &= \frac{Q_0\Delta}{ku_1}\frac{1}{l_0}\frac{dl_0}{dx}\eta\phi_1'(\eta) \\ &= \frac{Q_0\Delta}{ku_1}\frac{\log l_0/z_0 - 2\int_0^\infty \phi_1 \log\eta\, d\eta}{\log l_0/z_0 + 1 - 2\int_0^\infty \phi_1\log\eta\, d\eta}\frac{1}{x}\eta\phi_1'(\eta). \end{aligned} \tag{7.3}$$

The distribution function for the line source $\psi_1(\eta) = \eta\phi_1'(\eta)$ satisfies the conditions

$$\int_0^\infty \psi_1(\eta)\, d\eta = \tfrac{1}{2}, \quad \psi_1(0) = K_m/K_h, \tag{7.4}$$

and it seems certain that the temperature (or concentration) must decrease monotonically with height. These conditions restrict considerably the possible forms of $\psi_1(\eta)$, and, in particular, the centroid of the distribution cannot be at a height less than $\frac{1}{4}(K_h/K_m)\, l_0$, attained with the 'top-hat' distribution

$$\psi_1(\eta) = K_m/K_h \quad \text{for} \quad \eta < \tfrac{1}{2}K_h/K_m. \tag{7.5}$$

With the mixing-length assumption,

$$\left.\begin{aligned} \phi_1(\eta) &= -\frac{K_m}{K_h}\int_{(K_m/K_h)\eta}^\infty \frac{e^{-2x}}{x}\, dx, \\ \psi_1(\eta) &= \frac{K_m}{K_h}\exp(-K_m/K_h\, 2\eta), \\ \bar{z} &= \tfrac{1}{2}l_0 K_h/K_m, \end{aligned}\right\} \tag{7.6}$$

and for the Panofsky & Townsend profile

$$\left.\begin{aligned} &\phi_1(\eta) = K_m/K_h\{\log(\eta K_m/K_h) + (1-\eta K_m/K_h)\} \quad \text{for} \quad \eta < K_h/K_m, \\ &\psi_1(\eta) = K_m/K_h(1-\eta K_m/K_h), \\ &\quad \bar{z} = \tfrac{1}{3}K_h/K_m\, l_0. \end{aligned}\right\} \tag{7.7}$$

The temperature scale of the distribution is very nearly proportional to x^{-1} for large values of $\log l_0/z_0$, as can be seen from the second form in equation (7.3). Defining the 'exponent of distance'

$$m_{cl} = \frac{d(\log\theta_1)}{d(\log x)} = \frac{d(\log T(0))}{d(\log x)},$$

it is easily shown that

$$-m_{cl} = 1 - \left(\log l_0/z_0 + 1 - 2\int_0^\infty \phi_1 \log\eta\, d\eta\right)^{-2}, \tag{7.8}$$

and is usually within a few per cent of -1.

Cermak (1963) assumed Lagrangian similarity of the diffusion process from a line source to obtain results that are similar in form to these ones, but an analysis of diffusion observations in terms of the values of m_{cl} led him to the conclusion that the constant b in his expression for the centroid of the distribution

$$bkx = \bar{z}(\log \bar{z}/z_0 - 1) - (1-b)(\log h/z_0 + h) \tag{7.9}$$

was about 0·1 and certainly less than 0·2 (h is the height of the source). It was shown in I that b is simply a function of the profile shape, explicitly,

$$b = k\frac{K_h}{K_m}T(0)\int_0^\infty z\,T(z)\,dz \Big/ \left\{\int_0^\infty T(z)\,dz\right\}^2, \tag{7.10}$$

and the profile shapes observed at Cardington and Porton (Pasquill 1962) indicate that $b = 0{\cdot}9k(K_h/K_m)$ which is certainly much larger than 0·1. The particular observations which appear to be consistent only with the low value of the constant were obtained in wind tunnels with finite height of release, and the interpretation might be questioned on two grounds—the validity of the correction for finite height in equation (7.9) and the validity of the assumption that the diffused material is confined to a small fraction of the total thickness of the layer. The last assumption seems to be in error for the observations of Malhotra (1962) which provide the best evidence for a value of b near 0·1, at least in terms of the present theory. With the quoted roughness length of $2{\cdot}5 \times 10^{-5}$ ft, the value of l_0 at 4·5 ft from the source can be calculated from (7.1) to be nearly 0·19 ft compared with a boundary-layer thickness of about 0·25 ft. The comparatively small value of m_{cl}, interpreted by Cermak as the result of finite height of the source, could arise with equal plausibility from substantial diffusion into the outer part of the boundary layer and consequent limiting of the increase of $\bar{z}$.

At this point, it may be of interest to show how the self-preserving distribution (7.3) can be used to calculate the diffusion from an elevated line source at a height h, small compared with the total thickness of the layer. It was explained in I that the process of differentiating self-preserving distributions with respect to x can be extended to distributions caused by line doublets, line quadrupoles and so on. The line doublet distribution is

$$T = \frac{Q_1\Delta^2}{ku_1}\left\{-\frac{d}{dx}\left(\frac{1}{l_0}\frac{dl_0}{dx}\right)\psi_1 + \left(\frac{1}{l_0}\frac{dl_0}{dx}\right)^2 \eta\psi_1'\right\}, \tag{7.11}$$

and the quadrupole distribution

$$T = \frac{Q_1\Delta^3}{ku_1}\left\{\frac{d^2}{dx^2}\left(\frac{1}{l_0}\frac{dl_0}{dx}\right)\psi_1 - \frac{3}{2}\frac{d}{dx}\left(\frac{1}{l_0}\frac{dl_0}{dx}\right)^2\eta\psi_1' + \left(\frac{1}{l_0}\frac{dl_0}{dx}\right)^3\eta\frac{d}{d\eta}(\eta\psi_1')\right\}. \tag{7.12}$$

For large values of $\log l_0/z_0$, $l_0^{-1}dl_0/dx = 1/x$ very nearly, and the distributions approximate to

$$\left.\begin{aligned} T &= \frac{Q_1\Delta^2}{ku_1}x^{-2}\psi_2(\eta), \quad T = \frac{Q_1\Delta^3}{ku_1}x^{-3}\psi_3(\eta), \\ \text{where}\quad \psi_2 &= \psi_1 + \eta\psi_1', \\ \psi_3 &= 2\psi_1 + 4\eta\psi_1' + \eta^2\psi_1''. \end{aligned}\right\} \tag{7.13}$$

These and distributions for higher-order sources can be used to construct special initial distributions of temperature. As a simple example, we use the Panofsky & Townsend profile (7.7) to construct the distribution functions:

$$\left.\begin{aligned} \psi_1 &= 1-\eta && \text{for} \quad \eta \leqslant 1, \\ \psi_2 &= 1-2\eta && \text{for} \quad \eta \leqslant 1, \\ \psi_3 &= 2-6\eta+\delta(\eta-1) && \text{for} \quad \eta \leqslant 1. \end{aligned}\right\} \tag{7.14}$$

Notice that ψ_3 includes a δ-function of unit strength at $\eta = 1$. A concentrated source at height l_0 is provided by the combination

$$\eta^2\psi_1'' = 2\psi_1 - 4\psi_2 + \psi_3 = \delta(\eta-1),$$

and the temperature distribution caused by injection of heat at rate $Q_0\Delta$ at height h is

$$T = \frac{Q_0\Delta}{ku_1}\frac{1}{x+x_0}\left\{1-\eta-\frac{2x}{x+x_0}(1-2\eta)+\left(\frac{x}{x+x_0}\right)^2(1-3\eta+\tfrac{1}{2}\delta(\eta-1))\right\} \quad \text{for} \quad \eta \leqslant 1, \tag{7.15}$$

where x_0 is the distance from the source position to the origin of the self-preserving flows, given by

$$2k^2x_0 = h\left\{\log h/z_0 - 2\int_0^\infty \phi_1(\eta)\log\eta\, d\eta\right\}. \tag{7.16}$$

The temperature at ground level is

$$T(0) = \frac{Q_0\Delta}{ku_1}\frac{x^2}{(x+x_0)^3}, \tag{7.17}$$

and

$$-m_{cl} = 1 - \frac{3x_0}{x+x_0}. \tag{7.18}$$

Notice that the centroid of the line-source component is originally at a height of $\frac{1}{3}h$ and not at h as assumed by Cermak.

In general, the value of the exponent of distance is not sensitive to the value of the constant b and it is better to compare the theory with values of the depth of the diffused layer. Some relevant observations are quoted by Pasquill (1962). They refer to measurements downwind of a continuous source of pollutant and are presented as exponent of distance, height Z_0 at which the concentration has fallen to 10% of the ground concentration and the variance

$$Z^2 = \int_0^\infty cz^2 dz \Big/ \int_0^\infty c\, dz.$$

For the Panofsky & Townsend profile, $Z^2 = \frac{1}{6}l_0^2$ and $Z_0 = 0{\cdot}9l_0$, while for the mixing length profile, $Z^2 = \frac{1}{2}l_0^2$ and $Z_0 = 1{\cdot}15l_0$. The observed profiles are intermediate between these forms. In table 5, calculated values of l_0 and m_{cl} are compared with the observations and the agreement is good for $Z_0 = l_0$ and $Z^2 = \frac{1}{5}l_0^2$, roughly the mean of the values for the mixing-length and Panofsky & Townsend profiles.

In conclusion, the effects of instability on the diffusion are easily found by the method of §5. For example, the ground concentration remains proportional to $l_0^{-1} dl_0/dx$ and approximately

$$m_{cl} = -1 - \tfrac{1}{4}\alpha l_0/L + (\log l_0/z_0 - \tfrac{5}{2})^{-2}. \tag{7.19}$$

Observations	z_0 (cm)	l_0	$l_0/5^{\frac{1}{2}}$	$-m_{cl}$	Observed Z_0	Observed Z	Observed $-m_{cl}$
Porton (Pasquill 1962)	3	10·1	4·51	0·95	10	—	1·0, 0·9, 0·98
Cramer (1957)	< 1	7·0–8·0 (for $z_0 = 0·5$ and 1·0 cm)	3·1–3·6	0·97	—	3·5	≈ 1·0
Kazanskii & Monin (1957)	0·4	6·8	2·8	0·98	7 (visible top)	—	≈ 1·0
Cardington (229 m) (Pasquill 1962)	3	19	—	—	20 (strong winds)	—	—

N.B. heights are in metres.

TABLE 5. Diffusion parameters at 100 m from a linear source at ground level

8. Concluding remarks

Good agreement of theoretical predictions with the available observations has been found for flow after a change of roughness (Panofsky & Townsend 1964), for flow downwind of a fence, for temperature and humidity downwind of a change in surface, and for diffusion from a line-source. In most of these situations, the only disposable parameter is the shape of the relevant profile, whose variation is severely limited by the requirement that it assumes the logarithmic form inside the surface equilibrium layer. Consequently, little variation in the predicted magnitude of the more important quantities is found if the three profile shapes are interchanged. The observational evidence suggests that the real profiles are probably intermediate between the mixing-length profile and the Panofsky & Townsend (log-linear) profile. For the change-of-roughness flow, it may resemble the profile defined by equation (A 5) of paper I.

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